Les cavaliers de la tuile carrée

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LIFO, Université d'Orléans

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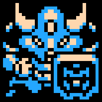


In this talk

We meet:

- tilings by Wang tiles
- deterministic tilesets
- simulations of Turing machines
- drawings of a diagonal

We introduce a new **simple syntactic model** to deal with a generalized notion of determinism and we present some simple *ad-hoc* **constructions**.

1. Tilings

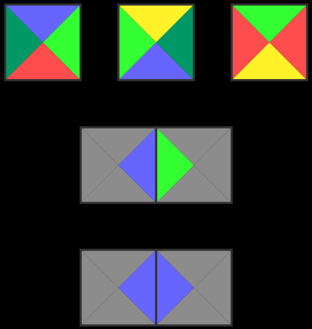
Tilings by Wang tiles

A **Wang tile** is an oriented (no rotations allowed) unit square tile carrying a **color on each side**.

A **tileset** τ is a finite set of Wang tiles.

A **configuration** $c \in \tau^{\mathbb{Z}^2}$ associates a tile to each cell of the discrete plane $\mathbb{Z}^2$.

A **tiling** is a configuration where the colors of the common sides of neighboring tiles match.



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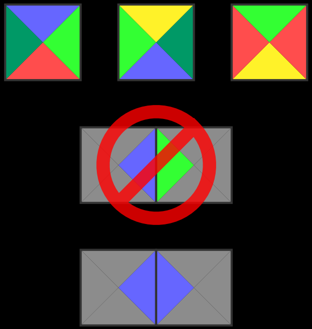
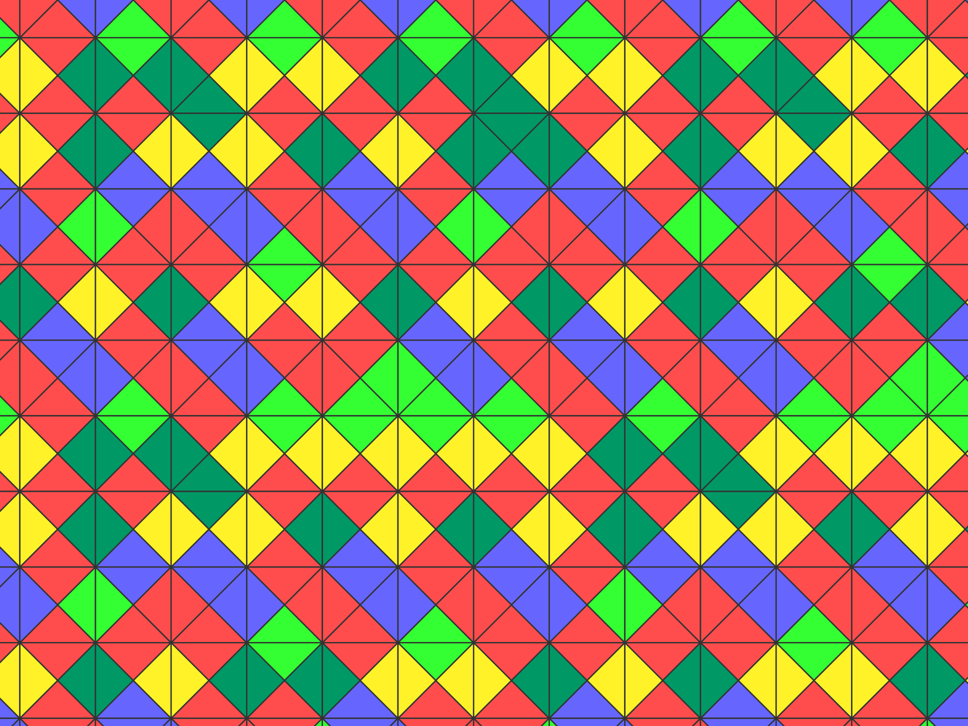


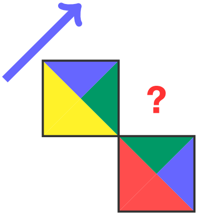


Deterministic tilesets

Introduced by J. Kari in 1991 to prove the undecidability of the nilpotency problem for 1D cellular automata.

Notations: *NW* for North-West, *SE* pour South-East…

Deterministic tileset. A tileset τ is **NE-deterministic** if for any pair of tiles $(t_W, t_S) \in \tau^2$, there exists **at most one** tile t compatible to the west with t_W and to the south with t_S.



Partial local map.

$$f : \tau^2 \to \tau$$

We symmetrically define **{NW,SE,SW}-determinism**.

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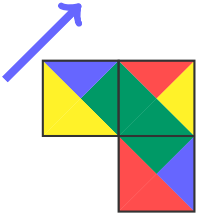

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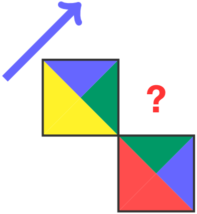
Deterministic tilesets

Bideterminism. A tileset is **bideterministic** if it is simultaneously deterministic in **two opposite directions**: NE & SW, or NW & SE.



4-way determinism. A tileset is **4-way deterministic** if it is simultaneously deterministic in the **4 directions** NE, NW, SW and SE.

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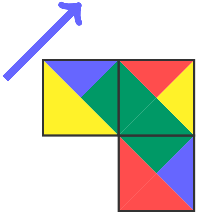

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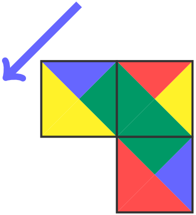

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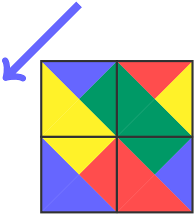

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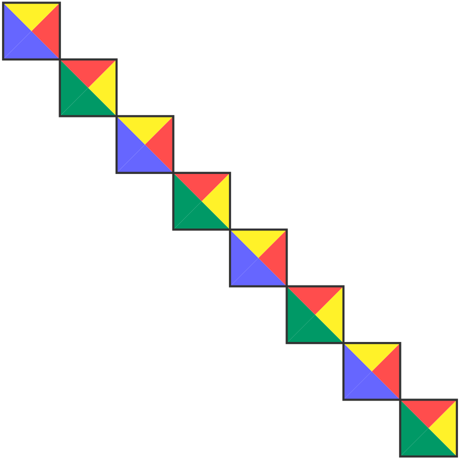

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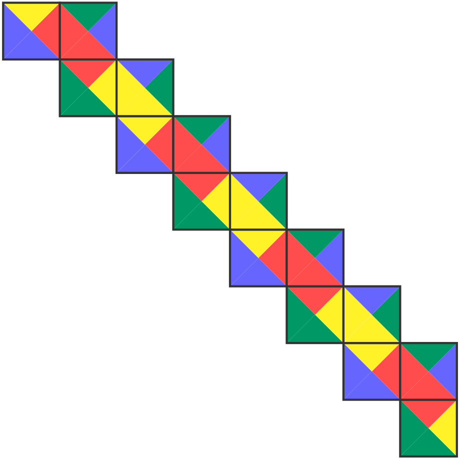

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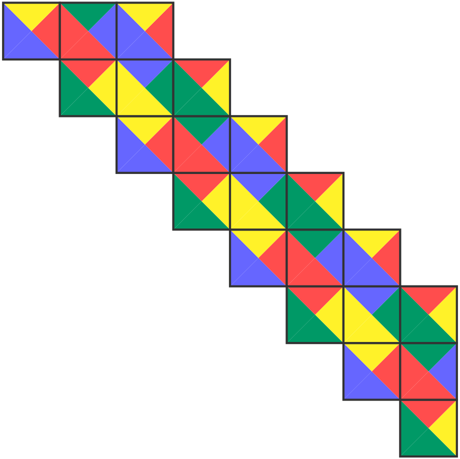

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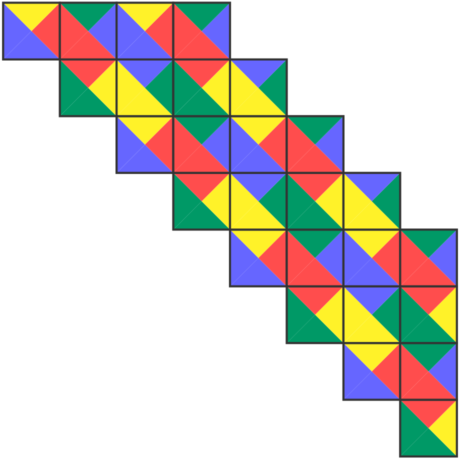

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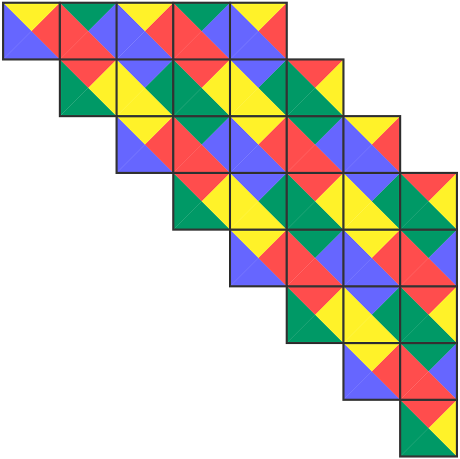

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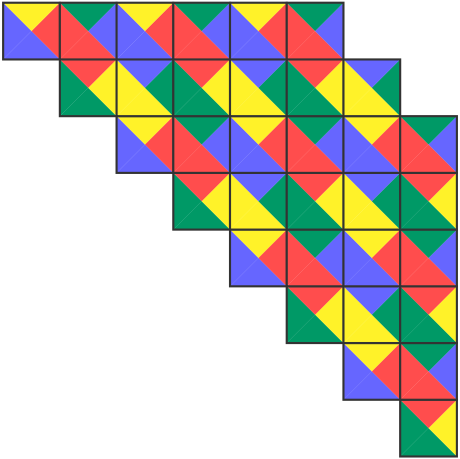

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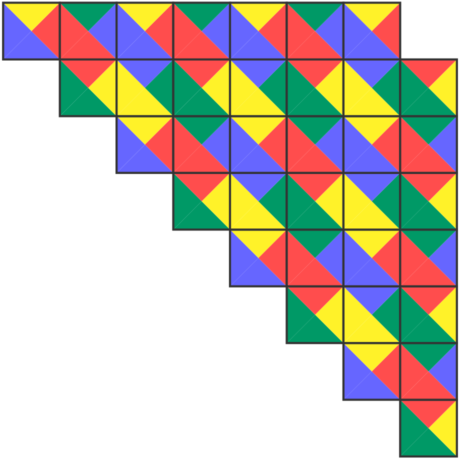

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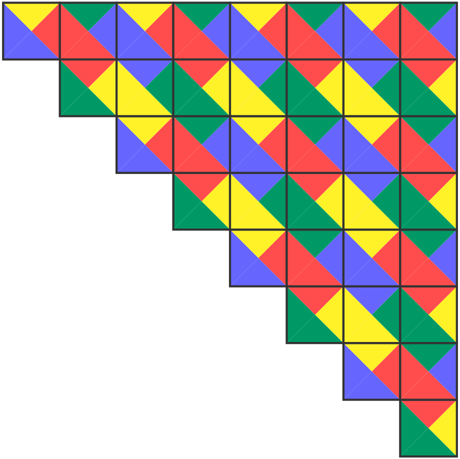

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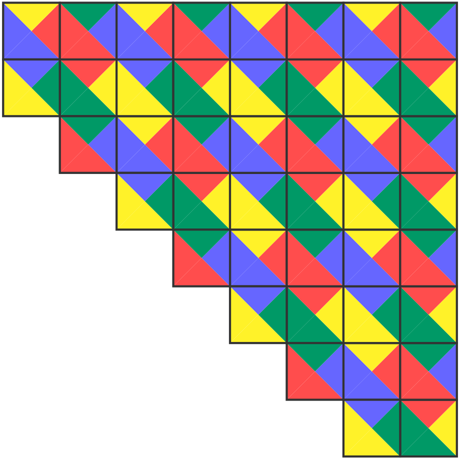

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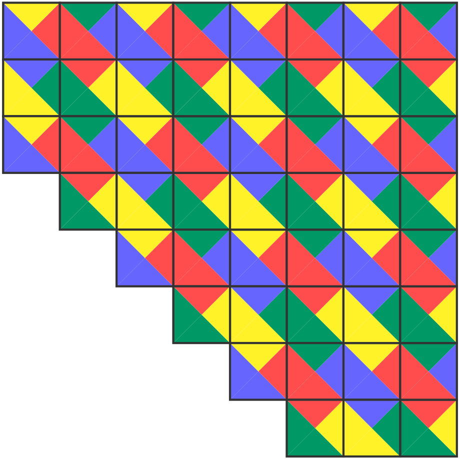

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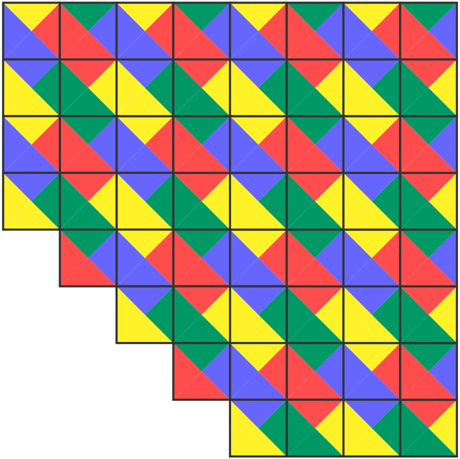

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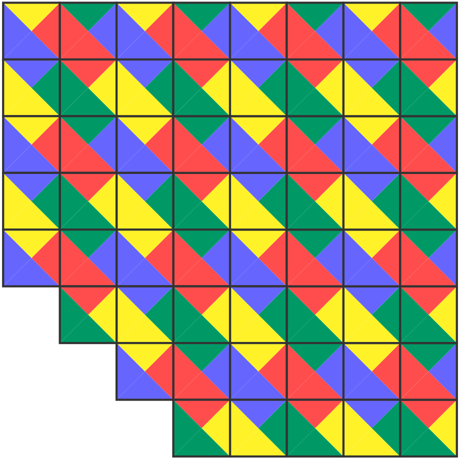

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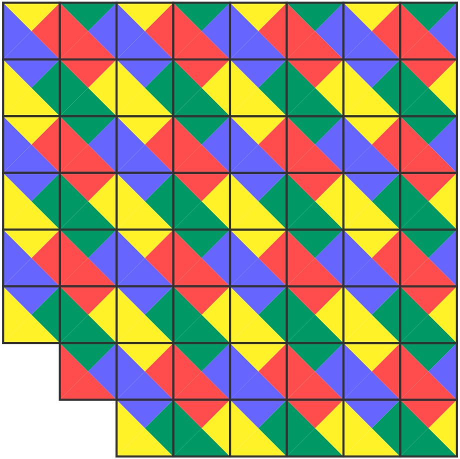

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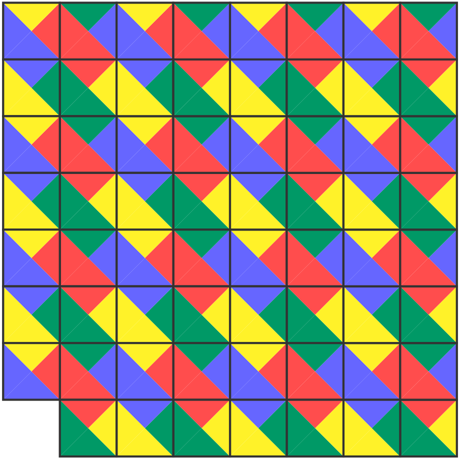

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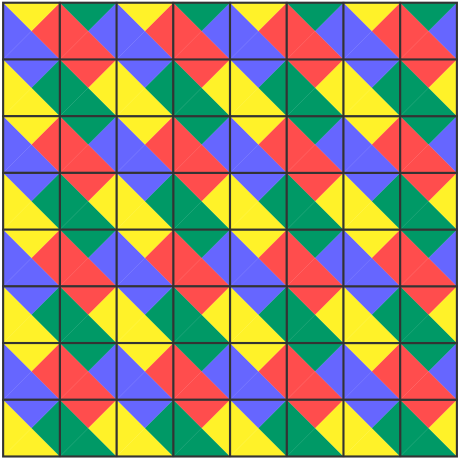

Deterministic tilesets: a short history

[Kari, 1991] introduced a (bi)determinization of **[Robinson, 1971]** to treat the nilpotency problem for cellular automata in dimension 1 (**Nil1D**).

Theorem [Kari, 1991]. **Nil1D** is undecidable.

Theorem [Kari, 1991]. There exist some **(bi)deterministic** aperiodic tilesets.

N.B. The 16 Wang tiles derived from Ammann's geometric tiles are bideterministic.

Theorem [Kari, 1991]. **DP** remains undecidable for **deterministic** tilesets.

Deterministic tilesets: a revised history

[Aanderaa-Lewis, 1974] embedded the coding of two-dimensional Wang tilings into one-dimensional double shifts, allowing this to be coded back into deterministic Wang tilesets or one-dimensional cellular automata.

Theorem [Aanderaa-Lewis, 1974]. **Nil1D** is undecidable.

Theorem [Aanderaa-Lewis, 1974]. There exist some **deterministic** aperiodic tilesets.

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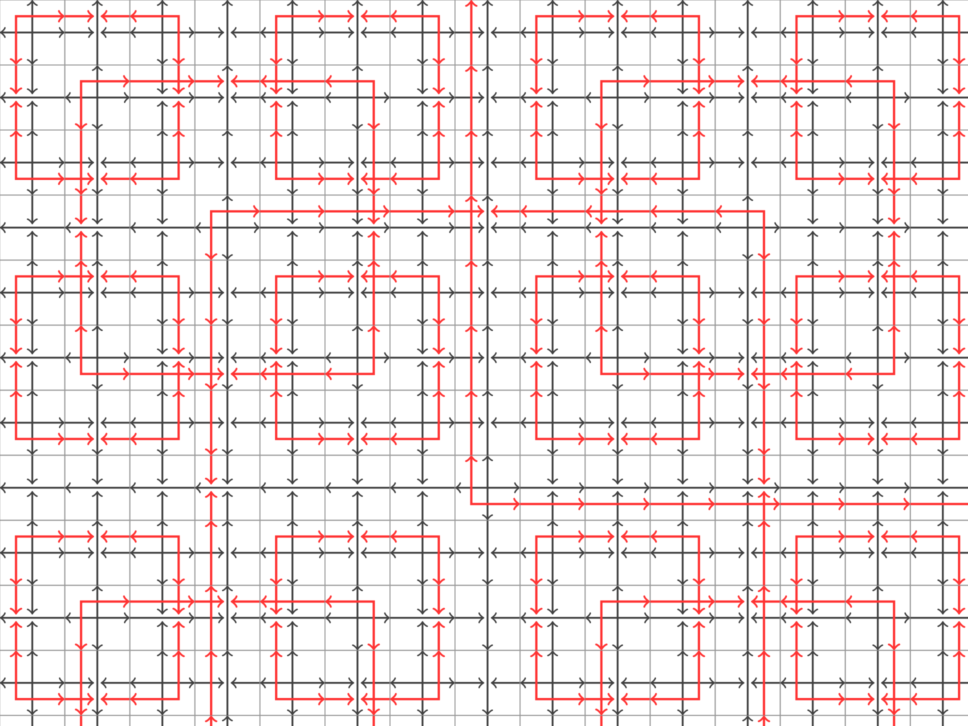
Theorem [Aanderaa-Lewis, 1974]. **DP** remains undecidable for **deterministic** tilesets.



Deterministic tilesets: a short history

[Kari-Papasoglu, 1999] builds a strong determinization of **[Robinson, 1971]**.

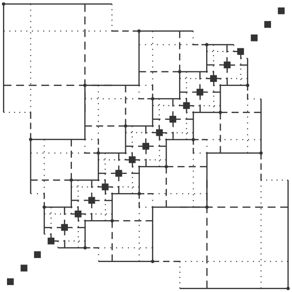
Theorem [Kari-Papasoglu, 1999]. There exist some **strongly deterministic** aperiodic tilesets.

[Lukkarila, 2009] introduces a 4-way determinization of **[Robinson, 1971] + Turing computation**.

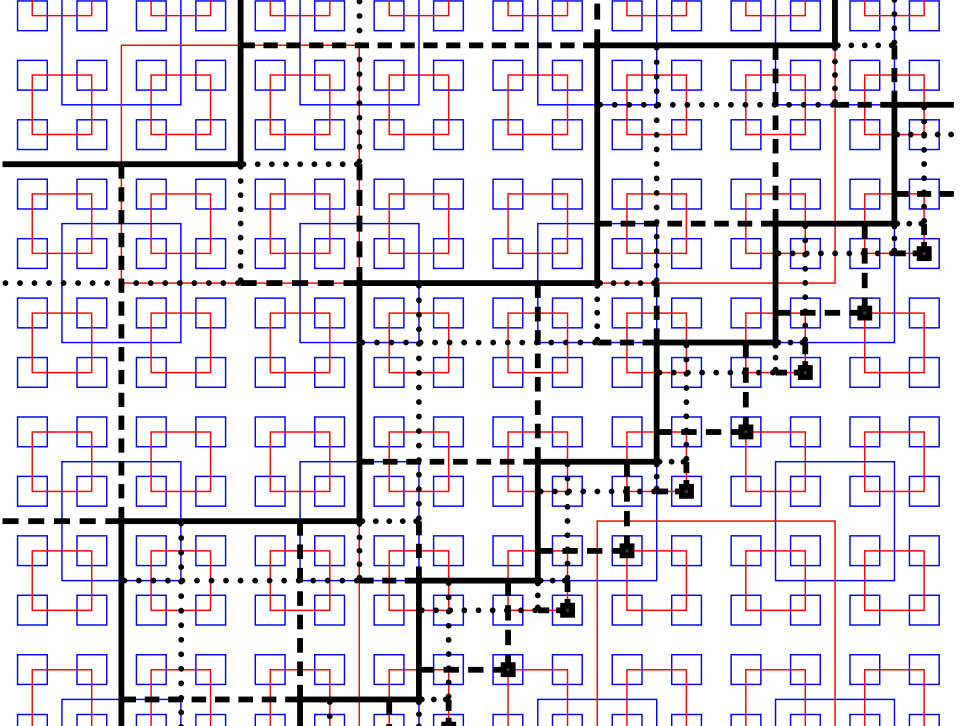
Theorem [Lukkarila, 2009]. **DP** remains undecidable for **strongly deterministic** tilesets.

Lukkarila’s diagonal

To deal with the zig-zag described by the moves of the head of a Turing machine, Lukkarila builds a tile set to mark a single diagonal: infinite *firing squad* over Robinson’s tile set.



Open problem [Lukkarila, 2009]. *Could there be a significantly simple tile set for drawing a single diagonal line 4-way deterministically?*



2. From Wang tiles to knight tiles

Basic ideas

Idea. To simplify the structure used by Lukkarila, one could use signals of several slopes to build an infinite firing squad structure.

Problem. 4-way deterministic Wang tiles can only be used to **locally** build **horizontal** and **vertical** signals. Any other signal has **angles** that cannot be seen locally by the local determinism rule without embedding the signal in a more **global** structure.

Proposition (*informal*). A "locally realizable" slope cannot be a direction of expansiveness.

Possible solution. Extending the **radius** of the local rule of determinism allows to see further…

Radius of determinism: first attempt

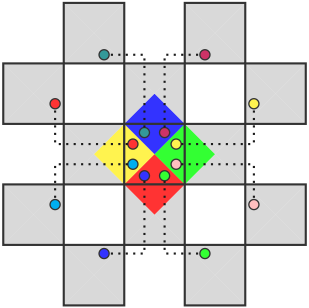
Radius of determinism. A tileset τ is *NE deterministic with radius r* if for all *valid* $(2r+1) \times (2r+1)$ square pattern by τ, the center tile is perfectly determined by the $2r$ tiles at positions $(1, 2r)$, $(2, 2r-1)$, ..., $(2r-1, 1)$.

The tileset is **4-way deterministic with radius** *r* if it is simultaneously deterministic with radius *r* in the 4 diagonal directions.

Proposition. If τ is a 4-way deterministic tileset with radius *r*, then $\mathcal{X}_\tau$ is (at least) expansive in directions $\left]-\frac{r}{r-1}, -\frac{r-1}{r}\right[\cup \left]\frac{r-1}{r}, \frac{r}{r-1}\right[$.

Knight tiles

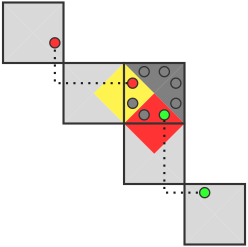
Idea. Introduce a convenient framework (inspired from PCA) to work with determinism at radius 2.



Knight tile. A *knight tile* is a Wang tile + 8 knight colors to be shared with each of the 8 tiles at a chess knight move distance of itself.

Knight determinism

Knight determinism. A knight tileset is *NE-deterministic* if it is uniquely determined by its (WN, W, S, SE) colors.



We symmetrically define *{NW,SW,SE}-determinism* and **4-way** determinism for knight tilesets.

Syntactic benefit. Determinism can directly be read on the tiles.

3. Back to Lukkarila’s I: the tiling problem with a seed tile

Reversible Turing machine

A **reversible Turing machine** is a 5-tuple $\left(\Sigma, Q, \overleftrightarrow{q_i}, F, \delta\right)$ where:

- Σ is the tape (finite) alphabet
- Q is the finite set of states
- the head always moves $\{\leftarrow, \rightarrow\}$ at each transition, we define $\overleftrightarrow{Q} = Q \times \{\leftarrow, \rightarrow\}$ and use the notations $\overrightarrow{q}$ (resp. $\overleftarrow{q}$) for $(q, \rightarrow)$ (resp. $(q, \leftarrow)$)
- the **partial injective** transition map $\delta : \overleftrightarrow{Q} \times \Sigma \rightarrow \overleftrightarrow{Q} \times \Sigma$
- $\overleftrightarrow{q_i} \in \overleftrightarrow{Q}$ is the initial state and $F \subset \overleftrightarrow{Q}$ is the set of final states (δ is not defined on F)

4-way simulation of a Turing machine

We start from a classical simulation of a Turing machine by Wang tiles.

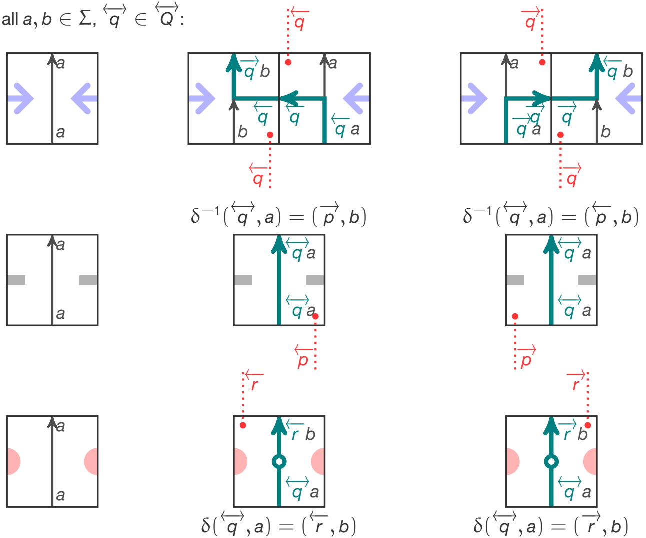
We must use **knight colors** to foresee the transition to come in every direction. We actually only use $\{NE, NW, SE, SW\}$ knight positions.

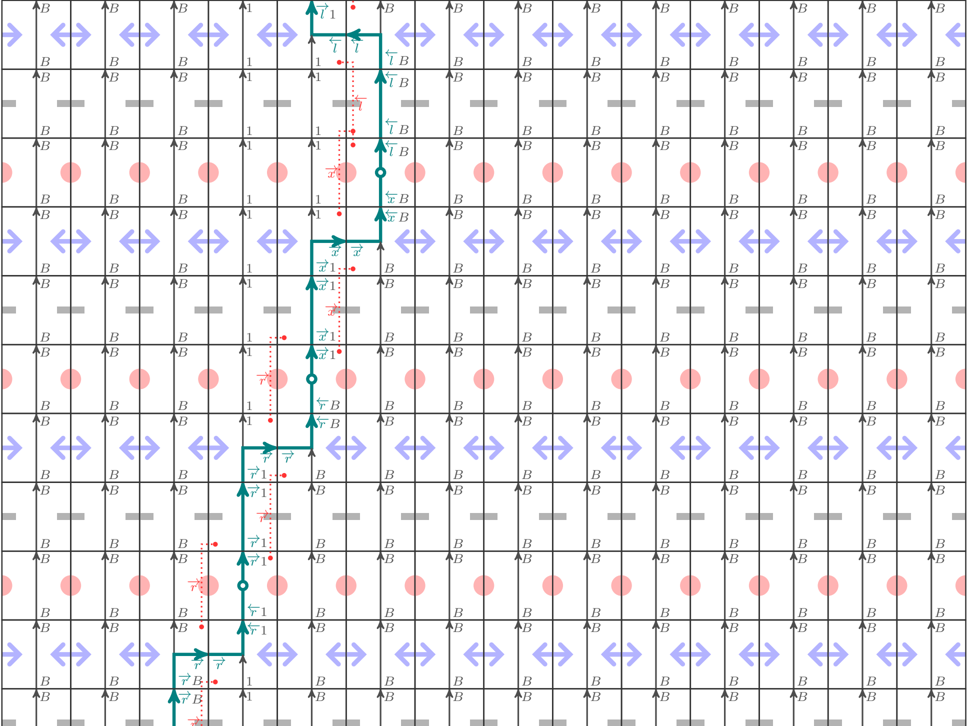
Due to the geometrical constraints of the knights, we slow down the computation by decomposing each transition into three phases:

- a **transition step** where only the state transition is done (but it codes for the future move of the head)
- a **waiting step** where nothing is done
- a **moving step** where the head is moved

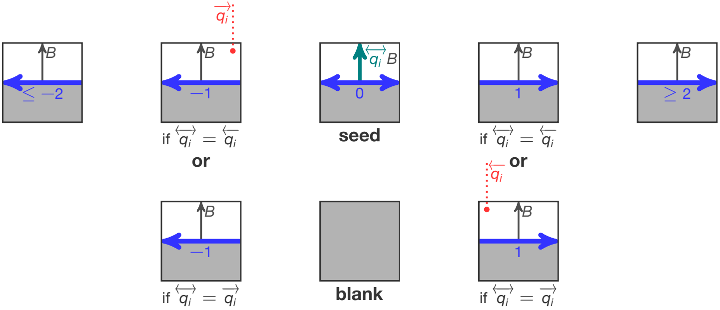
4-way simulation of a TM: the tiles

For all $a, b \in \Sigma$, $\overleftrightarrow{q} \in \overleftrightarrow{Q}$:





4-way simulation of a TM: initialization



Adding these **initialization tiles** + a simple technical layer (*omitted*) to ensure determinism, we get the following result.

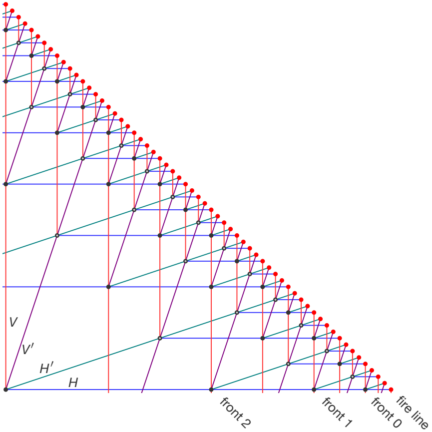
Theorem. The Domino Problem with a seed tile is undecidable for 4-way deterministic knight tilesets.

4. Back to Lukkarila’s II: drawing a diagonal

General structure (1/2)

Objective. Marking 4-way deterministically with knight tiles a single diagonal without relying on a complex aperiodic structure.

Idea. Use the general structure of a **firing squad**.



General structure (2/2)

The tiles of the **fire line** draws the diagonal and will be alphabetically projected onto **1**.

The other tiles are used for **construction signals** and will be alphabetically projected onto **0**.

The set of tilings will be projected onto the subshift generated by a configuration containing a **single diagonal of 1s** (and 0 everywhere else).

Technical aspects. We need to:

- ◇ locally assemble and disassemble the firing structure to ensure (classical Wang) determinism in NE and SW directions
- ◇ be able to locally complete any diagonal orthogonal to the firing line to ensure the (knight) determinism in NW and SE directions

NE+SW determinism: Thue-Morse (1/2)

Thue-Morse substitution. $s : \{0, 1\} \to \{0, 1\}^*$ defined by $s(0) = 01$ and $s(1) = 10$.

The **Thue-Morse word** is the unique **fixed-point** in $\{0, 1\}^{\mathbb{N}}$ of s starting with the letter 0.

0

By definition, the Thue-Morse word can be **unsubstituted** into itself by placing bars every two letters.

We similarly deal with bi-infinite ($\mathbb{Z}$ indexed) sequences over $\{0, 1\}$ whose factors all appear in the Thue-Morse sequence.

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NE+SW determinism: Thue-Morse (2/2)

A factor of the Thue-Morse word is **even** if it appears at a position where its first letter **directly follows a bar**. It is **odd** if it appears at a position where its first letter is **directly followed by a bar**.

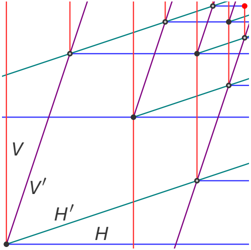
Lemma. Any factor of length at least 4 is either even or odd.

Moving a window of size 4 over the Thue-Morse word, we see an alternating sequence of even and odd factors.

Idea. Use this fact + the substitution/unsubstitution mechanism to build our structure deterministically.

Each **front line** contains a Thue-Morse sequence: a factor of size 4 by point. The sequence is **substituted** when approching the fire line and **unsubstituted** in the other direction.

NE+SW determinism: Structural rules

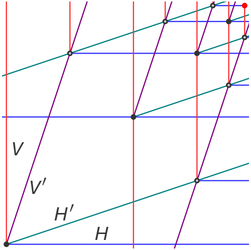


Each H/V (resp. H'/V') signal carries an **even** (resp. **odd**) word of size 4.

On **meeting points**:

- the 2 incoming signals carry the **same word** $u = a_1a_2a_3a_4$
- we **substitute** u to derive the words carried by the 4 outgoing signals: $s(a_1a_2a_3a_4) = \underbrace{b_1b_2b_3b_4}_{V} b_5b_6b_7b_8$
- this process is **reversible**!

NE+SW determinism: Structural rules



Each H/V (resp. H'/V') signal carries an **even** (resp. **odd**) word of size 4.

On **meeting points**:

- the 2 incoming signals carry the **same word** $u = a_1a_2a_3a_4$
- we **substitute** u to derive the words carried by the 4 outgoing signals: $s(a_1a_2a_3a_4) = b_1 \underbrace{b_2b_3b_4b_5}_{V'} b_6b_7b_8$
- this process is **reversible**!

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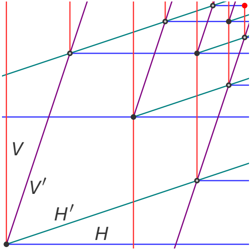

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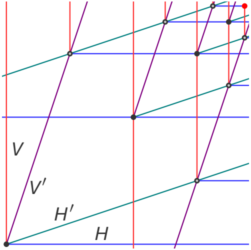

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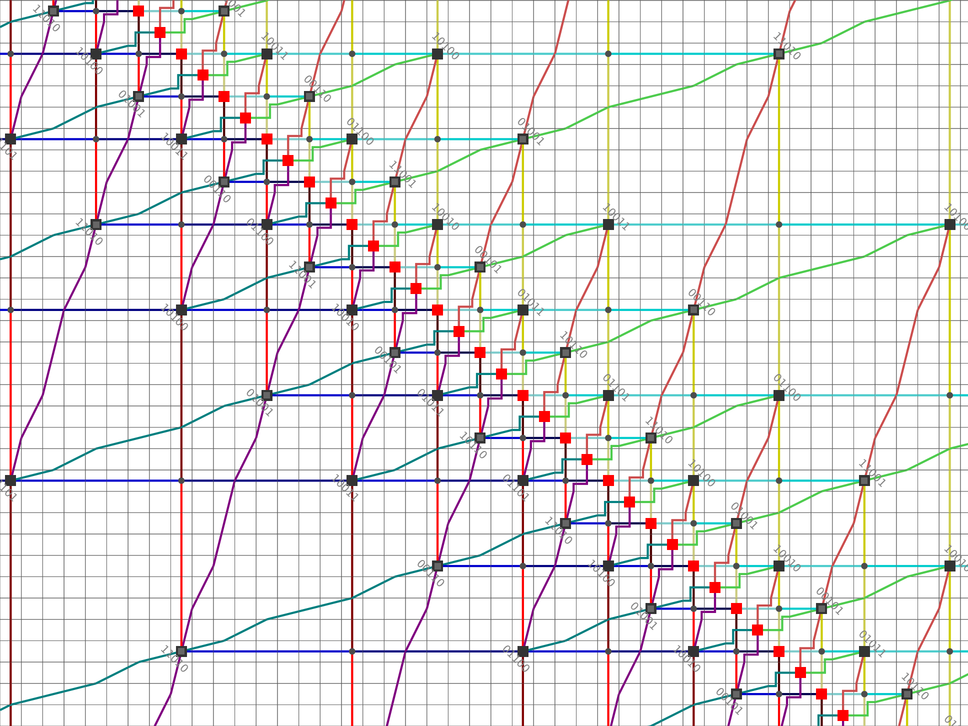
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NW+SE determinism

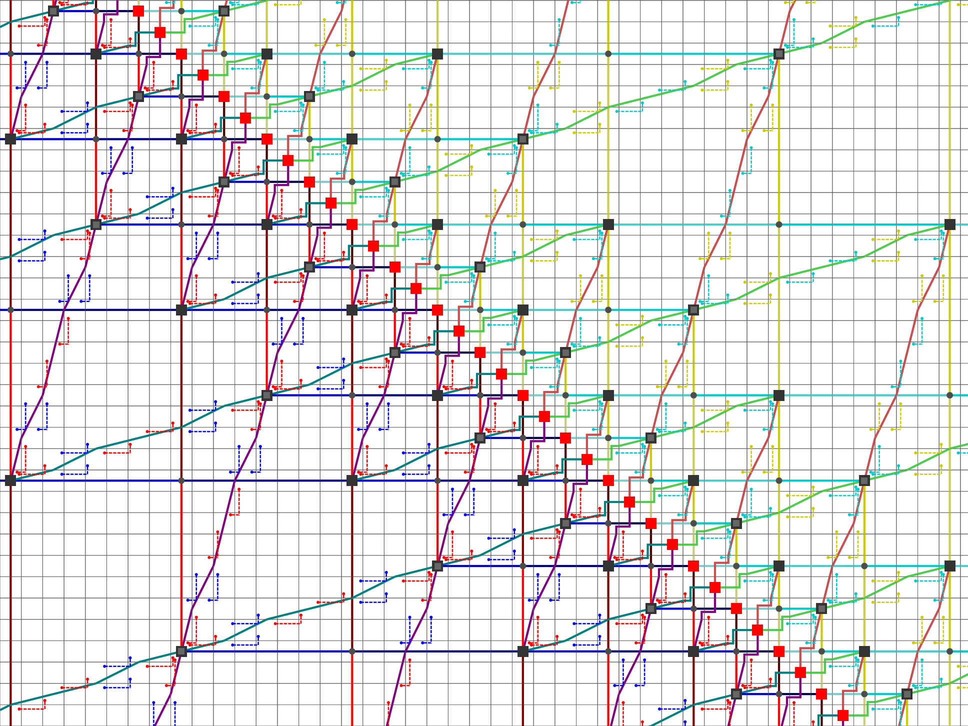
We use some {WS,SW,EN,NE} knight colors to:

- build deterministically the H' and V' signals…(easy!)
- … with the factors they carry (*should not be* that hard)

This is actually a bit tricky as:

- added knight colors must remain predictable in NE+SW directions…
- … hence **a knight cannot cross a front line**!

Technical solution. **Twisting** the H'/V' signals + technical **particular cases** for the fire line.



Putting everything together

The built knight tileset is **4-way deterministic** (classical Wang determinism in directions NE+SW) and is **not** aperiodic.

The tilings can be projected onto the single diagonal subshift.

Seen as a one-dimensional cellular automaton acting on the diagonals, it is **time-symmetric**: It is its own inverse up to a swap in colors.

The tilings are (at least) expansive in directions $]-\infty, 0[\cup]\frac{1}{2}, 2[$.

5. Conclusion

Conclusion and perspectives

◇ Simple and elegant **aperiodic knight tileset** to give a completely simplified proof of the undecidability of DP?

◇ Hierarchy on the **radiuses** of determinism?

◇ Clarify the link between **expansiveness**, some generalized notion of **determinism** and the “locally realizable” **slopes**.

That's all folks!

Thank you for your attention.