Computational complexity of majority automata under different updating schemes

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October 24, 2013

Automata network

An *Automata Network* is a triple $\mathcal{A} = (G, Q, f_i : i \in V)$, where

- $G = (V, E)$ is a simple undirected graph and $V = \{1, \ldots, n\}$.
- Q the set of states ($Q = \{0, 1\}$)
- $f_i : \{0, 1\}^n \to \{0, 1\}$ is the transition function associated to the vertex i.

We say that vertices in state 1 are *active* while vertices in state 0 are *passive*.

Updating Schemes

An *updating scheme* (US) of the automaton $\mathcal{A}$ is a function

$$\phi : V \to \{1 \ldots |V|\}$$

st. if u and v are vertices and $\phi(u) < \phi(v)$ then the state of u is updated before v, and if $\phi(u) = \phi(v)$ then nodes u and v are update at the same time.

- **Synchronous:** $\phi = 1$.
  (All vertices are updated at the same time.)
- **Sequential:** $\phi = \sigma$, where σ is a permutation of V.
  (One vertex at a time)
- **Block sequential:**

$$V = \cup_{i=1}^{k} V_i, \qquad \cap_{i=1}^{k} V_i = \emptyset, \qquad \phi|_{V_i} = i$$

  The vertex set is partitioned into several subsets, st. into the same set every vertex is updated at the same time, and different subsets are updated sequentially in some order.

Trajectory of a configuration

Let $x \in \{0,1\}^n$ be a configuration of an automaton. The trajectory $T^{\phi}(x)$ of x with the updating scheme ϕ is the set

$$T^{\phi}(x) = \{x(t) : t \geq 0\}$$

where $x(0) = x$ and $x(t+1)$ is obtained from $x(t)$ after every vertex is updated according to ϕ.

The trajectory of x *enters in a limit cycle of period p* if $|T(x(t))| = p$ for some $t \geq 0$.
(A cycle of period 1 is a *fixed point*.)

There are at most 2^n different configurations (finite graph), then the trajectory of any configuration eventually enters to a limit cycle for any US. (Steady state)

$\tau_\phi(x)$: steps to reach the steady state starting from x with a US ϕ.

$\tau_\phi(\mathcal{A}) = \max\{\tau_\phi(x) : x \in \{0,1\}^n\}$ is the transient length of $\mathcal{A}$.

Decision Problem

One Cell Prediction: OCP

Given:

- An automaton $\mathcal{A} = (G, \{0,1\}, (f_i : i \in V))$,
- $x \in \{0,1\}^n$ a configuration of $\mathcal{A}$,
- ϕ an updating scheme of $\mathcal{A}$,
- and $v \in V$ a vertex initially passive ($x_v = 0$),

Does there exists $y \in T^{\phi}(x)$ such that $y_v = 1$?

Majority automata

Here we will consider only *majority functions*, i.e.:

$$f_i(x) = \begin{cases} 1 & \text{if} \quad \sum_{j \in N(i)} x_i > \frac{|N(i)|}{2} \\ 0 & \text{if} \quad \sum_{j \in N(i)} x_i \leq \frac{|N(i)|}{2} \end{cases}$$

where $N(i)$ is the set of neighbors of vertex i.

An automata network with this rule is called a *majority automata*.

Parallel and sequential US.

Theorem

For parallel and sequential updating schemes, **OCP** *is in* **P**

Idea: Simulate $\mathcal{A}$ until v changes.

For a configuration $x(t)$

- For any $i \in V$, $x_i(t+1)$ can be computed in $\mathcal{O}(n)$ time.
- $x(t+1)$ can be computed in $\mathcal{O}(n^2)$ time.

$|T(x)| = |\{x(t) \ : \ t > 0\}|$ is *poly*(n)?

[E. Goles, F. Fogelman, D. Pellegrin]

If $\{f_1, \dots, f_n\}$ are threshold functions with weights matrix A and threshold vector b.

$$E_{syn}[x(t)] = -\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i(t-1)x_i(t) + \sum_{i=1}^{n} b_i(x_i(t) + x_i(t-1))$$

$$E_{seq}[x(t)] = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i(t)x_j(t) + \sum_{i\in V} b_ix_i(t)$$

[E. Goles, F. Fogelman, D. Pellegrin]

- $|E(x)|$ is $\mathcal{O}(n^2)$
- $\Delta_t E = E[x(t+1)] - E[x(t)] \leq 0$
  (E constant in cycles)
- Synchronous US $\rightarrow$ reach at most cycles of length 2.
  Sequential US $\rightarrow$ reach only fixed points.
- $\tau(\mathcal{A})$ is $\mathcal{O}(n^3)$

Block sequential US

Theorem

There is a block sequential update scheme in a majority automata, such that each block has cardinality 2 and the limit cycle has a super-polynomial length.

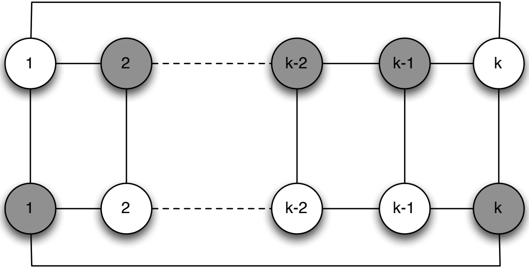
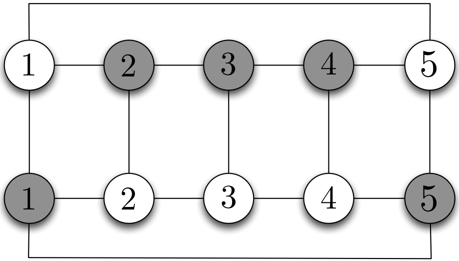
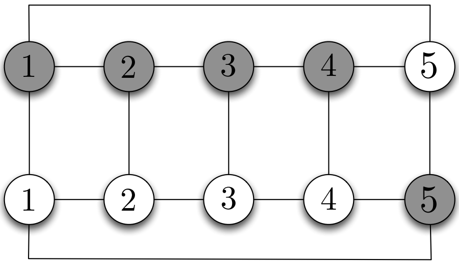


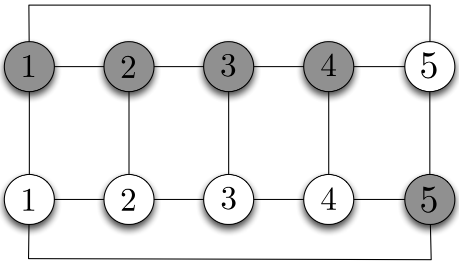
Figure: Ladder



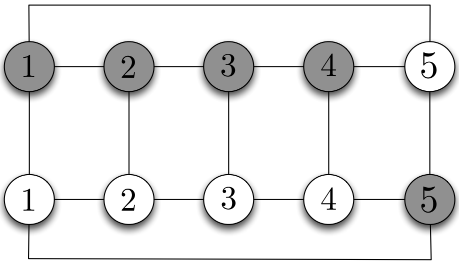
$t = 0$



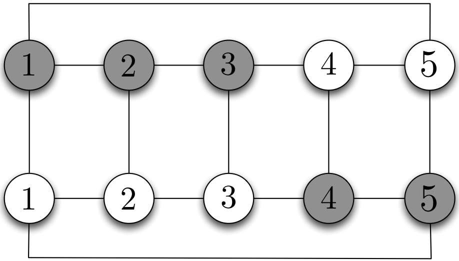
$t = 0 - 1$



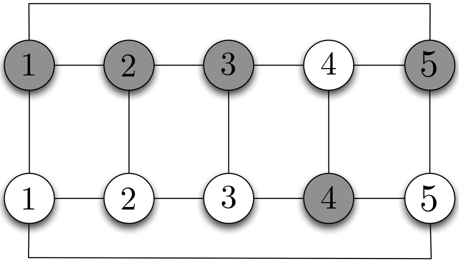
$t = 0 - 2$



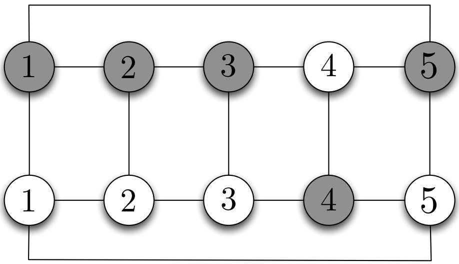
$t = 0 - 3$



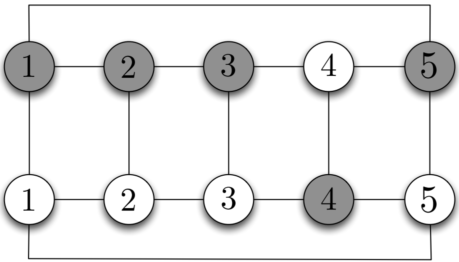
$t = 0 - 4$



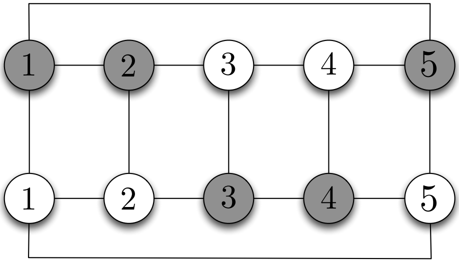
$$t = 0 - 5 = 1$$



$$t = 1 - 1$$



$t = 1 - 2$



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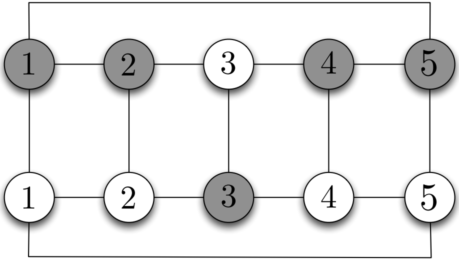
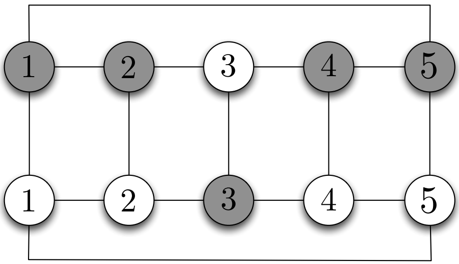
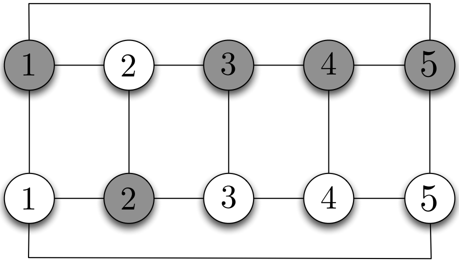


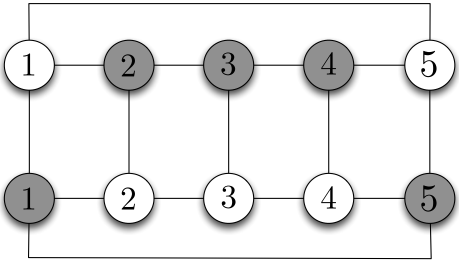
$t = 1 - 4$



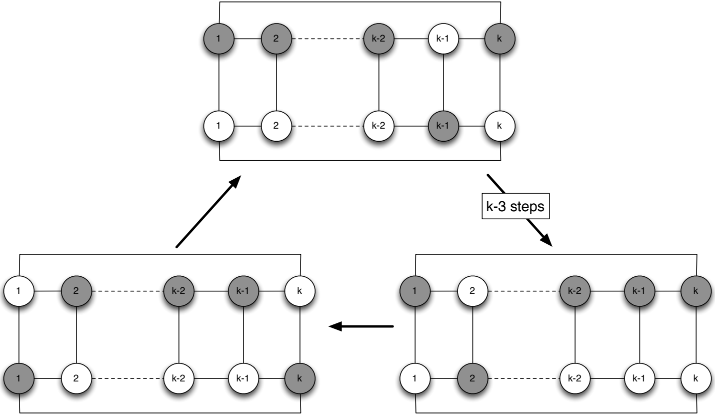
$$t = 1 - 5 = 2$$



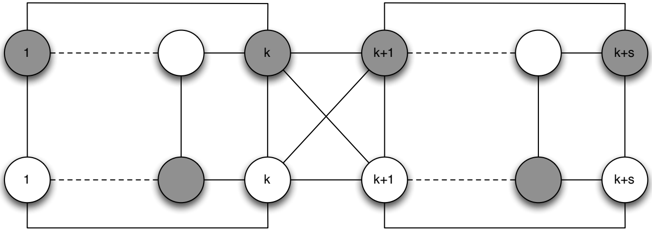
$t = 3$



$t = 4$



Limit cycle of length $k-1$



Limit cycle of length $lcm(k-1, s-1)$

Let m be a positive integer, and let $\pi(m)$ the number of primes not exceeding m.

Let G the graph obtained from $\pi(m)$ ladders of sizes $(p_1+1),(p_2+1),\dots,(p_{\pi(m)}+1)$, where $\{p_1,p_2\dots,p_{\pi(m)}\}$ the first $\pi(m)$ primes.

Then

$$V(G) \leq \sum_{i=1}^{\pi(m)} 2(p_i+1) \leq 2\pi(m)(m+1)$$

$$\text{limit cycle of } G = lcm(p_1,\dots,p_{\pi(m)}) = \prod_{i=1}^{\pi(m)} p_i = e^{\theta(m)}$$

where $\theta(m) = \sum_{i=1}^{\pi(m)} \log(pi)$.

From the Prime Number Theorem:

$$lcm(p_1,\dots,p_{\pi(m)}) \geq e^{\Omega(\sqrt{|V(G)|\log(|V(G)|)})}$$

For block sequential update schemes...

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OCP is in **P**?

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Clearly **OCP** is in **P**-SPACE.

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Theorem

The problem **OCP** *is NP-Hard for block sequential updating schemes.*

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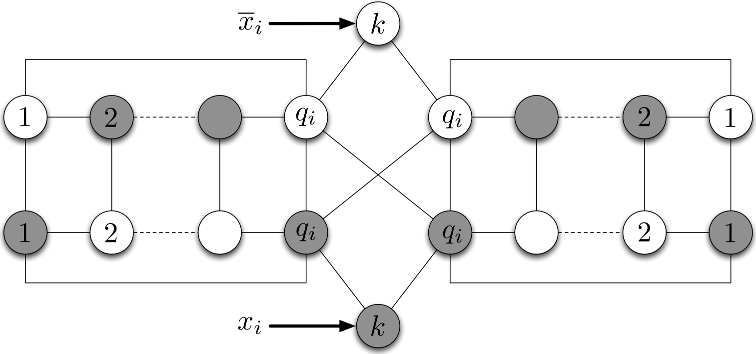
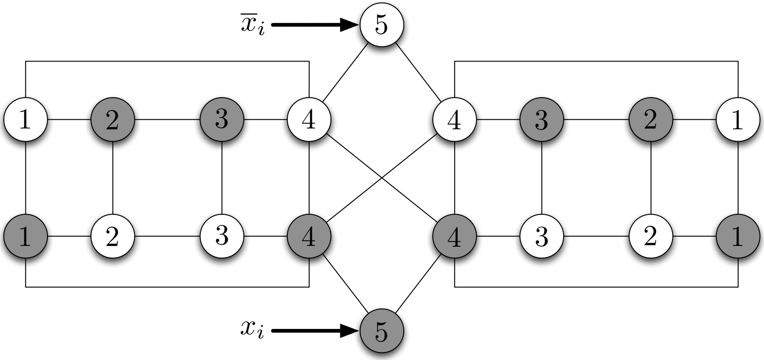
Theorem

The problem **OCP** *is NP-Hard for block sequential updating schemes.*

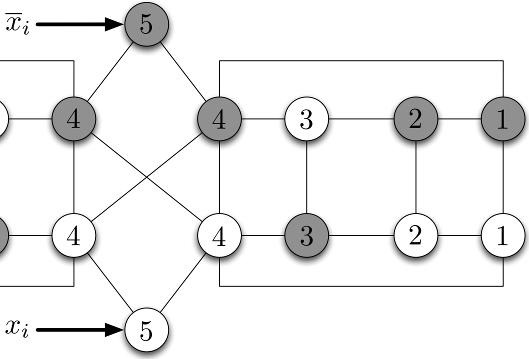
Proof: Reduce $3 - SAT$.
Let φ a 3CFN formula.



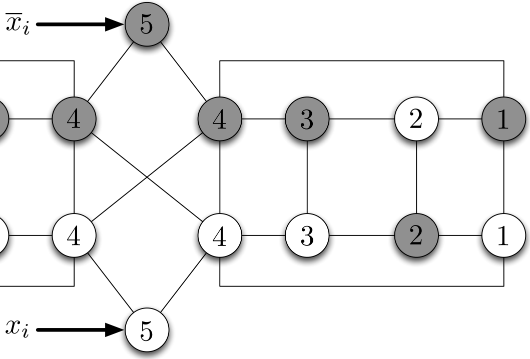
Gadget for variable x_i (positive and negative literals)



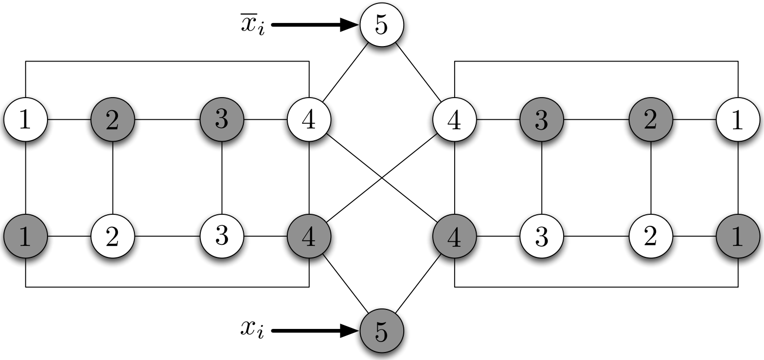
$t = 0$



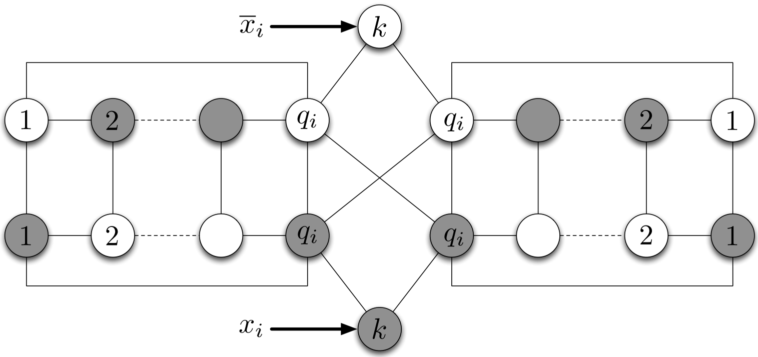
$t = 1$



$t = 2$



$t = 3$



$q_i = p_i + 1$ where p_i i-th prime

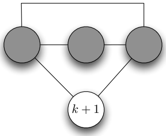
$k = p_n + 2$

$x_i = 1$ in steps multiple of p_i, and $x_i = 0$ otherwise

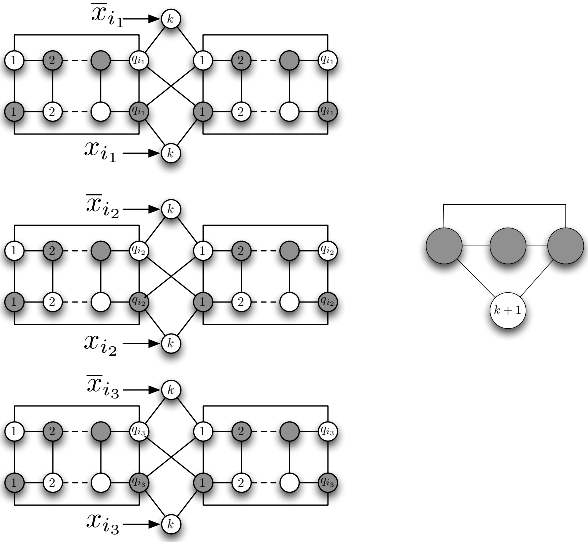
Then, any possible input value $x = (x_1, x_2, x_3, \ldots, x_n)$ of φ, $x_i \in \{0, 1\}$ come out in step

$$2^{x_1} 3^{x_2} 5^{x_3} \ldots p_n^{x_n}$$

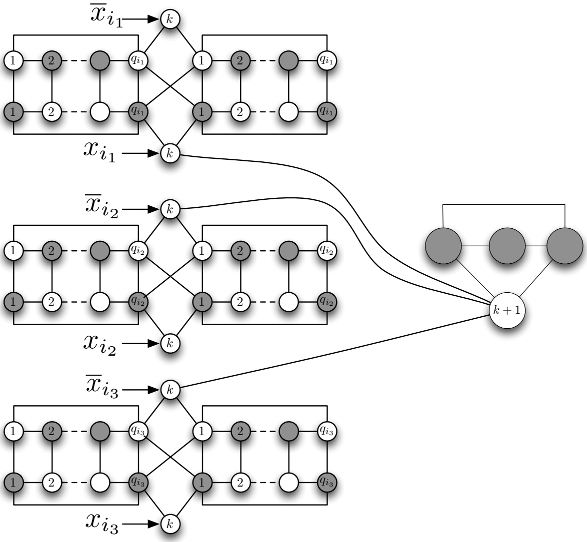
$$C_i = (x_{i_1} \vee \overline{x}_{i_2} \vee \overline{x}_{i_3})$$

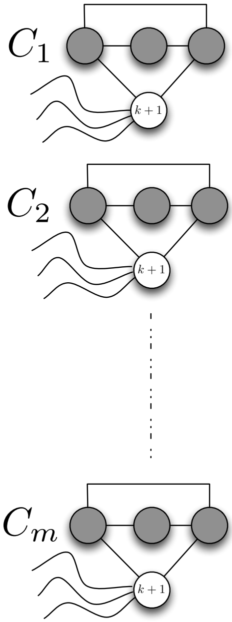


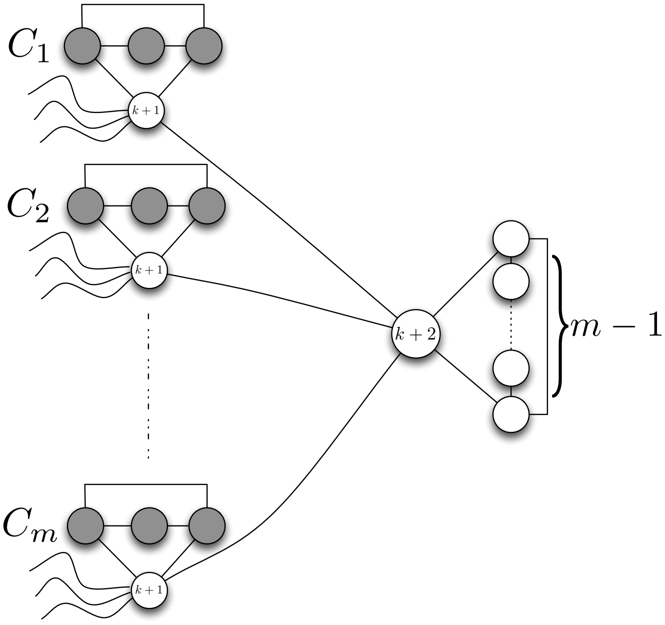
$$C_i = (x_{i_1} \vee \overline{x}_{i_2} \vee \overline{x}_{i_3})$$



$$C_i = (x_{i_1} \vee \overline{x}_{i_2} \vee \overline{x}_{i_3})$$




C_1
$k+1$
C_2
$k+1$
C_m
$k+1$


C_1
$k+1$
C_2
$k+1$
C_m
$k+1$
$k+2$
$m-1$

Conclusions

For the majority automata:

- For synchronous and sequential US, **OCP** is in P.
  (is P-Complete)
- For the block sequential updating schemes the problem is NP-Hard.
- [Goles, Montealegre, Salo, Törmä]: PSPACE-Completeness.

Future Work

- Constant the number blocks: constant length of limit cycles?
- Block sequential US over special families of graphs
- Other rules